

On 1-cocycles induced by a positive definite function on a locally compact abelian group

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Abstract

For φ a normalized positive definite function on a locally compact abelian group G , we consider on the one hand the unitary representation π_φ associated to φ by the GNS construction, on the other hand the probability measure μ_φ on the Pontryagin dual \hat{G} provided by Bochner's theorem. We give necessary and sufficient conditions for the vanishing of 1-cohomology $H^1(G, \pi_\varphi)$ and reduced 1-cohomology $\overline{H}^1(G, \pi_\varphi)$. For example, $\overline{H}^1(G, \pi_\varphi) = 0$ if and only if either $\text{Hom}(G, \mathbb{C}) = 0$ or $\mu_\varphi(1_G) = 0$, where 1_G is the trivial character of G .

1 Introduction

The Gel'fand-Naimark-Segal construction (see [BHV08]) provides a correspondence between positive definite functions φ on a locally compact group G and cyclic representations π_φ on Hilbert space. This allows one to construct a dictionary between the functional-analytic and algebro-geometric pictures of φ and π_φ . For example, φ is an extreme point in the cone $\mathcal{P}(G)$ of positive definite functions on G if and only if π_φ is an irreducible representation; or, there exists a constant $a > 0$ such that $\varphi - a$ is again positive definite if and only if π_φ has nonzero fixed vectors (see [Dix69]).

In view of their importance for rigidity questions and Kazhdan's property (T)¹, it is natural to try to fit 1-cohomology and reduced 1-cohomology of π_φ in that dictionary. This is the question we address in this paper, assuming G to be a locally compact abelian group. Indeed, in this case we enjoy Bochner's Theorem (see [Fol95]): φ is the Fourier transform of a positive Borel measure μ_φ on the Pontryagin dual \hat{G} . Without relying on the cohomological machinery available in the literature (see [Gui80, BHV08]), we achieve by completely elementary means the results of this paper, namely, that the

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¹Recall Shalom's results, see Theorems 0.2 and 6.1 in [Sha00]: for a compactly generated group G , the group G has property (T), if and only if $\overline{H}^1(G, \pi) = 0$ for every unitary representation π of G , if and only if $H^1(G, \sigma) = 0$ for every unitary irreducible representation of G .

existence of nontrivial 1-cohomology is determined by two factors: the existence of non-trivial homomorphisms to \mathbb{C} , and more importantly, the behavior of μ_φ near the trivial character 1_G .

2 Statement of results

For G a locally compact group and π a unitary representation of G on a Hilbert space \mathcal{H}_π , recall that the space of *1-cocycles* for π is

$$Z^1(G, \pi) = \{b : G \rightarrow \mathcal{H}_\pi : b \text{ continuous, } b(gh) = \pi(g)b(h) + b(g) \text{ for all } g, h \in G\}.$$

The space of *1-coboundaries* for π is:

$$B^1(G, \pi) = \{b \in Z^1(G, \pi) : \exists v \in \mathcal{H}_\pi \text{ such that } b(g) = \pi(g)v - v \text{ for every } g \in G\}.$$

The *1-cohomology* of π is then the quotient

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi).$$

Endow $Z^1(G, \pi)$ with the topology of uniform convergence on compact subsets of G . The *reduced 1-cohomology* of π is the quotient of the space of 1-cocycles by the closure of the space of 1-coboundaries, i.e.

$$\overline{H}^1(G, \pi) = Z^1(G, \pi)/\overline{B^1(G, \pi)}.$$

From now on, let G be a locally compact *abelian* group, φ a positive definite function on G . Excluding the zero function, we may without loss of generality take φ to be normalized ($\varphi(e) = 1$). Let μ_φ be the probability measure on the Pontryagin dual \hat{G} provided by Bochner's theorem, i.e. $\varphi(x) = \int_{\hat{G}} \xi(x) d\mu_\varphi(\xi)$ for $x \in G$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the cyclic representation of G associated to φ through the GNS construction, so that the cyclic vector $\xi_\varphi \in \mathcal{H}_\varphi$ satisfies $\langle \pi_\varphi(x)\xi_\varphi | \xi_\varphi \rangle = \varphi(x)$. Let also ρ_φ be the representation of G on $L^2(\hat{G}, \mu_\varphi)$ given by $(\rho_\varphi(x)f)(\xi) = \xi(x)f(\xi)$ ($\xi \in \hat{G}, f \in L^2(\hat{G}, \mu_\varphi)$).

If λ_G denotes the regular representation of G and $h \in L^2(G)$, then $(\widehat{\lambda_G(x)h})(\xi) = \overline{\xi(x)h(\xi)}$. From Plancherel's Theorem, it follows that the composition of the Fourier transform with conjugation is a unitary equivalence between the regular representation on $L^2(G)$ and the unitary representation defined by $(f(\xi) \mapsto \xi(x)f(\xi))$ on $L^2(\hat{G})$. This, together with Bochner's Theorem, intuitively introduces ρ_φ , as well as the following proposition which we prove in section 3.

Proposition 1 *The representations π_φ and ρ_φ are unitarily equivalent.*

We assume from now on that φ is not the constant function 1, so that μ_φ is not the Dirac mass at the trivial character 1_G of G . This is still equivalent to $\mu_\varphi(1_G) < 1$. Let μ_φ^\perp be the probability measure on \hat{G} defined by $\mu_\varphi = \mu_\varphi(1_G)\delta_{1_G} + (1 - \mu_\varphi(1_G))\mu_\varphi^\perp$.

Let π_φ^0 be the (trivial) subrepresentation of π_φ on the subspace \mathcal{H}_φ^0 of π_φ -fixed vectors, and π_φ^\perp be the subrepresentation on the orthogonal complement, so that $\pi_\varphi = \pi_\varphi^0 \oplus \pi_\varphi^\perp$. A simple computation in the ρ_φ -picture shows that $\mathcal{H}_\varphi^0 \neq 0$ if and only if $\mu_\varphi(1_G) > 0$, and in this case $\mathcal{H}_\varphi^0 = \mathbb{C}\delta_{1_G}$. Moreover the map $L^2(\hat{G} \setminus \{1_G\}, \mu_\varphi^\perp) \rightarrow L^2(\hat{G}, \mu_\varphi) : f \mapsto \frac{f}{\sqrt{1 - \mu_\varphi(1_G)}}$ is isometric and identifies π_φ^\perp with the restriction of ρ_φ to $L^2(\hat{G} \setminus \{1_G\}, \mu_\varphi^\perp)$.

Our main result is:

Theorem 1 *Let φ be a nonconstant, normalized positive definite function on a locally compact abelian group G .*

1) *Consider the following statements:*

- i) $H^1(G, \pi_\varphi) = 0$;
- ii) *Both of the following properties are satisfied:*
 - a) $\mu_\varphi(1_G) = 0$ or $\text{Hom}(G, \mathbb{C}) = 0$;
 - b) $1_G \notin \text{supp}(\mu_\varphi^\perp)$.

Then (ii) \Rightarrow (i), and the converse holds if G is σ -compact.

2) *The following are equivalent:*

- i) $\overline{H}^1(G, \pi_\varphi) = 0$;
- ii) $\mu_\varphi(1_G) = 0$ or $\text{Hom}(G, \mathbb{C}) = 0$.

This result will be proved in section 4. It is essentially equivalent to Theorem 4 in [Gui72], but we emphasize the fact that our proof is direct and based on explicit construction of cocycles and coboundaries.

3 Proof of Proposition 1

Lemma 1 *The constant function $1 \in L^2(\hat{G}, \mu_\varphi)$ is a cyclic vector for ρ_φ .*

Proof: For $f \in L^1(G)$, consider the operator $\rho_\varphi(f) = \int_G f(x) \rho_\varphi(x) dx$; then $(\rho_\varphi(f).1)(\xi) = \int_G f(x) \xi(x) dx = \hat{f}(\xi)$.

Denote by $C_0(\hat{G})$ the space of continuous functions vanishing at infinity on \hat{G} , and recall that $\hat{f} \in C_0(\hat{G})$ (the Riemann-Lebesgue Lemma). It is classical that the map $L^1(G) \rightarrow C_0(\hat{G})$ is a continuous algebra homomorphism with dense image (a consequence of Stone-Weierstrass). Now compose this homomorphism with the continuous inclusion $C_0(\hat{G}) \rightarrow L^2(\hat{G}, \mu_\varphi) : h \mapsto h.1$. Since continuous functions with compact support are dense in $L^2(\hat{G}, \mu_\varphi)$, this inclusion has dense image. Since the map $L^1(G) \rightarrow L^2(\hat{G}, \mu_\varphi) : f \mapsto \rho_\varphi(f).1$ is the composite of the previous maps, it has dense image, meaning that 1 is cyclic for ρ_φ . ■

Observe that $\langle \rho_\varphi(x).1 | 1 \rangle = \int_{\hat{G}} \xi(x) d\mu_\varphi(\xi) = \varphi(x)$, so Proposition 1 follows from Lemma 1 and the uniqueness statement of the GNS construction. ■

4 Proof of Theorem 1

Since $\pi_\varphi = \pi_\varphi^0 \oplus \pi_\varphi^\perp$, we have $H^1(G, \pi_\varphi) = H^1(G, \pi_\varphi^0) \oplus H^1(G, \pi_\varphi^\perp)$ and analogously for \overline{H}^1 . As $B^1(G, \pi_\varphi^0) = 0$, we see that $H^1(G, \pi_\varphi^0) = 0$ if and only if $\overline{H}^1(G, \pi_\varphi^0) = 0$, if and only if either $\mu_\varphi(1_G) = 0$ or $\text{Hom}(G, \mathbb{C}) = 0$: this proves the implications $(i) \Rightarrow (ii)(a)$ in part 1 of Theorem 1, and $(i) \Rightarrow (ii)$ in part 2 of Theorem 1; moreover, it reduces the main result to:

Theorem 2 *Let φ be a nonconstant, normalized positive definite function on a locally compact abelian group G .*

- 1) *If $1_G \notin \text{supp}(\mu_\varphi^\perp)$, then $H^1(G, \mu_\varphi^\perp) = 0$. The converse holds if G is σ -compact.*
- 2) *$\overline{H}^1(G, \pi_\varphi^\perp) = 0$.*

Example 1 *In Part 1 of Theorems 1 and 2, the converse implications are false when G is not assumed to be σ -compact. Indeed, let G be an uncountable abelian group with the discrete topology, and take $\varphi = \delta_1$. Then π_φ is the left regular representation λ_G on $\ell^2(G)$, while $\mu_\varphi = \mu_\varphi^\perp$ is the Haar measure on the compact group \hat{G} . Since μ_φ has full support, in particular 1_G lies in its support. On the other hand $H^1(G, \lambda_G) = 0$ by Proposition 4.13 in [CTV08]*

To prove the implication “ \Rightarrow ” in part 1 of Theorem 2, we will need:

Lemma 2 *Let F be a closed subset of \hat{G} , with $1_G \notin F$.*

- a) *There exists a regular Borel probability measure ν_0 on G such that the Fourier transform $\widehat{\nu}_0$ vanishes on F .*
- b) *For every $\varepsilon > 0$, there exists a compactly supported regular Borel probability measure ν on G such that $|\widehat{\nu}| < \varepsilon$ on F .*

Proof: (a) See section 1.5.2 in [Rud62].

(b) Let ν_0 be a probability measure on G as in (a). Let δ be a number $0 < \delta < 1$, to be determined later. Let C be a compact subset of G such that $\nu_0(C) > 1 - \delta$. Let ν be the probability measure on G defined by $\nu(B) = \frac{\nu_0(B \cap C)}{\nu_0(C)}$, for every Borel subset $B \subseteq G$. By taking δ small enough, the total variation distance $|\nu_0 - \nu|(G)$ between ν_0 and ν can be made arbitrarily small. For any finite signed measure μ on G , we have the classical inequality $|\int_G f(x) d\mu(x)| \leq \|f\|_\infty |\mu|(G)$; applied to $\mu = \nu_0 - \nu$ and $f(x) = \xi(x)$ with $\xi \in \hat{G}$, it gives $|\widehat{\nu}_0(\xi) - \widehat{\nu}(\xi)| \leq |\nu_0 - \nu|(G)$, so that $\|\widehat{\nu}_0 - \widehat{\nu}\|_\infty < \varepsilon$ for δ small enough. ■

Proof of “ \Rightarrow ” in part 1 of Theorem 2: We assume that 1_G is not in the support of μ_φ^\perp , and prove that $H^1(G, \pi_\varphi^\perp) = 0$. Let $b \in Z^1(G, \pi_\varphi^\perp)$ be a 1-cocycle. Expanding $b(xy) = b(yx)$ using the cocycle relation, we get:

$$(1 - \pi_\varphi^\perp(x))b(y) = (1 - \pi_\varphi^\perp(y))b(x) \quad (x, y \in G).$$

In the realization of π_φ^\perp on $L^2(\hat{G}, \mu_\varphi^\perp)$, this gives:

$$(1 - \xi(x))b(y)(\xi) = (1 - \xi(y))b(x)(\xi) \quad (1)$$

almost everywhere in ξ (w.r.t. μ_φ^\perp). By Lemma 2, we can find a compactly supported probability measure ν on G such that $|1 - \hat{\nu}| \geq \frac{1}{2}$ on $\text{supp}(\mu_\varphi^\perp)$. Define an element $v \in L^2(\hat{G}, \mu_\varphi^\perp)$ by $v := \int_G b(y) d\nu(y)$: since b is continuous and ν is compactly supported, the integral exists (in the weak sense) in $L^2(\hat{G}, \mu_\varphi^\perp)$. Integrating (1) w.r.t. ν in the variable y , we get:

$$(1 - \xi(x))v(\xi) = (1 - \hat{\nu}(\xi))b(x)(\xi) \quad (2)$$

almost everywhere in ξ . Since $|1 - \hat{\nu}| \geq \frac{1}{2}$ on $\text{supp}(\mu_\varphi^\perp)$, the function $w(\xi) := \frac{v(\xi)}{1 - \hat{\nu}(\xi)}$ belongs to $L^2(\hat{G}, \mu_\varphi^\perp)$, and by (2) its coboundary is exactly b . ■

Lemma 3 *Let H be a locally compact group. Let $(\sigma_n)_{n \geq 1}$ be a sequence of unitary representations of H without nonzero fixed vectors, with σ_n acting on a Hilbert space \mathcal{H}_n . Assume that, for each $n \geq 1$, there exists a unit vector $\eta_n \in \mathcal{H}_n$ such that the series $\sum_{n=1}^\infty \|\sigma_n(x)\eta_n - \eta_n\|^2$ converges uniformly on compact subsets of H . Set $\sigma = \oplus_{n=1}^\infty \sigma_n$. Then $b(x) := \oplus_{n=1}^\infty (\sigma_n(x)\eta_n - \eta_n)$ defines a nonzero element in $H^1(G, \sigma)$.*

Proof: By assumption $b(x)$ belongs to $\oplus_{n=1}^\infty \mathcal{H}_n$ and the map $H \rightarrow \oplus_{n=1}^\infty \mathcal{H}_n : x \mapsto b(x)$ is continuous. Let $\eta \in \prod_{n=1}^\infty \mathcal{H}_n$ be defined as $\eta = (\eta_n)_{n \geq 1}$. Since b is the formal coboundary of η , we have $b \in Z^1(H, \sigma)$. To prove that b is not a coboundary, it suffices to show that the associated affine action $\alpha(x)v = \sigma(x)v + b(x)$ on $\oplus_{n=1}^\infty \mathcal{H}_n$ has no fixed point. But $\alpha(x)v = v$ translates into $\sigma_n(x)(v_n + \eta_n) = v_n + \eta_n$ for every $x \in H$ and $n \geq 1$. Since σ_n has no nonzero fixed vector, we have $v_n + \eta_n = 0$ so $\|v_n\| = 1$, which contradicts $\sum_{n=1}^\infty \|v_n\|^2 < +\infty$. ■

Proof of “ \Leftarrow ” in part 1 of Theorem 2, assuming G to be σ -compact: Let $(K_n)_{n \geq 0}$ be an increasing sequence of compact subsets of G , with $G = \bigcup_{n=1}^\infty K_n$, and $K_0 = \{1\}$. Define a basis $(U_k)_{k \geq 0}$ of open neighborhoods of 1_G in \hat{G} by $U_k = \{\xi \in \hat{G} : \max_{g \in K_k} |\xi(g) - 1| < 2^{-k}\}$ (observe that $U_0 = \hat{G}$). Define a sequence $(k_n)_{n \geq 0}$ inductively by $k_0 = 0$ and $k_n = \min\{k : k > k_{n-1}, \mu_\varphi^\perp(U_k) < \mu_\varphi^\perp(U_{k_{n-1}})\}$ for $n \geq 1$ (since $\mu_\varphi^\perp\{1_G\} = 0$ and 1_G is in the support of μ_φ^\perp , this is well-defined). Set then $C_n := U_{k_n} \setminus U_{k_{n+1}}$ for $n \geq 1$, and let \mathcal{H}_n be the space of functions in $L^2(\hat{G}, \mu_\varphi^\perp)$ which are μ_φ^\perp -almost everywhere 0 on $\hat{G} \setminus C_n$. Then \mathcal{H}_n is a closed, ρ_φ -invariant subspace of $L^2(\hat{G}, \mu_\varphi^\perp)$. Denote by σ_n the restriction of ρ_φ to \mathcal{H}_n , so that $L^2(\hat{G}, \mu_\varphi^\perp) = \oplus_{n=0}^\infty \mathcal{H}_n$ and $\rho_\varphi = \oplus_{n=0}^\infty \sigma_n$. Let $\eta_n = \frac{1_{C_n}}{\sqrt{\mu_\varphi^\perp(C_n)}}$ be the normalized characteristic function of C_n . To appeal to Lemma 3, we still have to check that $x \mapsto \sum_{n=1}^\infty \|\sigma_n(x)\eta_n - \eta_n\|^2$ converges uniformly on every compact subset K of G . Clearly we may assume $K = K_\ell$. For $n \geq \ell$ and $x \in K_\ell$ and $\xi \in C_n$, we have $|\xi(x) - 1| < 2^{-k_n}$, hence:

$$\max_{x \in K_\ell} \sum_{n=\ell}^\infty \|\sigma_n(x)\eta_n - \eta_n\|^2 = \max_{x \in K_\ell} \sum_{n=\ell}^\infty \frac{1}{\mu_\varphi^\perp(C_n)} \int_{C_n} |\xi(x) - 1|^2 d\mu_\varphi^\perp(\xi) \leq \sum_{n=\ell}^\infty 4^{-k_n} \leq \sum_{n=0}^\infty 4^{-n} = \frac{4}{3}$$

and

$$\max_{x \in K_\ell} \sum_{n=0}^{\infty} \|\sigma(x)\eta_n - \eta_n\|^2 \leq \left(\max_{x \in K_\ell} \sum_{n=0}^{\ell-1} \|\sigma(x)\eta_n - \eta_n\|^2 \right) + \frac{4}{3} \leq 4\ell + \frac{4}{3} < +\infty.$$

So the result follows from Lemma 3. ■

Proof of part 2 of Theorem 2: Let $b \in Z^1(G, \pi_\varphi^\perp)$ be a 1-cocycle. We must show that b is a limit of 1-coboundaries (uniformly on compact subsets of G). Since $\mu_\varphi^\perp(1_G) = 0$, by regularity of μ_φ^\perp , we may find a decreasing sequence of relatively compact open neighborhoods $(V_n)_{n \geq 1}$ of 1_G , such that $\mu_\varphi^\perp(V_n) \rightarrow 0$ for $n \rightarrow \infty$. Set $\mathcal{H}_n := \{f \in L^2(\hat{G}, \mu_\varphi^\perp) : f = 0 \text{ a.e. on } V_n\}$; then \mathcal{H}_n is a closed, ρ_φ -invariant subspace, and the sequence $(\mathcal{H}_n)_{n \geq 1}$ is increasing with dense union in $L^2(\hat{G}, \mu_\varphi^\perp)$. Let ρ_n denote the restriction of ρ_φ to \mathcal{H}_n , and b_n be the projection of b onto \mathcal{H}_n . Then $b_n \in Z^1(G, \rho_n)$, and $\lim_{n \rightarrow \infty} b_n = b$ (uniformly on compact subsets of G). But since 1_G does not belong to the closed subset $\hat{G} \setminus V_n$, by part 1 of Theorem 2 we have $H^1(G, \rho_n) = 0$, so that b_n is a coboundary. ■

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